A DEFINABLE HENSELIAN VALUATION WITH HIGH QUANTIFIER COMPLEXITY

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Abstract. We give an example of a parameter-free definable henselian valuation ring which is neither definable by a parameter-free $\forall\exists$-formula nor by a parameter-free $\exists\forall$-formula in the language of rings. This answers a question of Prestel.

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1. Introduction

There have been several recent results concerning definitions of henselian valuation rings in the language of rings, mostly using formulae of low quantifier complexity (see [CDLM13], [Hon14], [AK14], [Feh15], [JK15], [Pre15], [FP15] and [FJ15]). After a number of these results had been proven, Prestel showed a Beth-like Characterization Theorem which gives criteria for the existence of low-quantifier definitions for henselian valuations:

Theorem 1.1 ([Pre15, Characterization Theorem]). *Let Σ be a first order axiom system in the ring language $\mathcal{L}_{\text{ring}}$ together with a unary predicate $\mathcal{O}$. Then there exists an $\mathcal{L}_{\text{ring}}$-formula $\phi(x)$, defining uniformly in every model $(K, \mathcal{O})$ of Σ the set $\mathcal{O}$, of quantifier type*

$$\begin{aligned}
\exists &\ \textit{iff}\ (K_1 \subseteq K_2 \Rightarrow \mathcal{O}_1 \subseteq \mathcal{O}_2)\\
\forall &\ \textit{iff}\ (K_1 \subseteq K_2 \Rightarrow \mathcal{O}_2 \cap K_1 \subseteq \mathcal{O}_1)\\
\exists\forall &\ \textit{iff}\ (K_1 \prec_\exists K_2 \Rightarrow \mathcal{O}_1 \subseteq \mathcal{O}_2)\\
\forall\exists &\ \textit{iff}\ (K_1 \prec_\exists K_2 \Rightarrow \mathcal{O}_2 \cap K_1 \subseteq \mathcal{O}_1)
\end{aligned}$$

for all models $(K_1, \mathcal{O}_1)$, $(K_2, \mathcal{O}_2)$ of Σ. Here $K_1 \prec_\exists K_2$ means that K_1 is existentially closed in K_2, i.e., every existential $\mathcal{L}_{\text{ring}}$-formula $\rho(x_1, \ldots, x_m)$ with parameters from K_1 that holds in K_2 also holds in K_1.

Applying the conditions in Theorem 1.1, it is easy to see that most known parameter-free definitions of henselian valuation rings in $\mathcal{L}_{\text{ring}}$ are in fact equivalent to $\emptyset$-$\forall\exists$-formulae or $\emptyset$-$\exists\forall$-formulae. Consequently, Prestel asked the following:

Question 1.2. *Let (K, w) be a henselian valued field such that $\mathcal{O}_w$ is a $\emptyset$-definable subset of K in the language $\mathcal{L}_{\text{ring}}$. Is there already a $\emptyset$-$\forall\exists$-formula or a $\emptyset$-$\exists\forall$-formula which defines $\mathcal{O}_w$ in K?*

The aim of this note is to provide a counterexample to Prestel's question. More precisely, we show:

Theorem 1.3. *There are ordered abelian groups Γ_1 and Γ_2 such that for any PAC field k with $k \neq k^{\text{sep}}$ the henselian valuation ring $\mathcal{O}_w = k((\Gamma_1))[[\Gamma_2]]$ is $\emptyset$-definable in the field $K = k((\Gamma_1))((\Gamma_2))$. However, $\mathcal{O}_w$ is neither definable by a $\emptyset$-$\forall\exists$-formula nor by a $\emptyset$-$\exists\forall$-formula in K.*

Moreover, we consider a specific example, namely the case $k = \mathbb{Q}^{\text{tot}\mathbb{R}}(\sqrt{-1})$. Here, $\mathbb{Q}^{\text{tot}\mathbb{R}}$ denotes the totally real numbers, that is the maximal extension of $\mathbb{Q}$ such that

for every embedding of the field into the complex numbers the image lies inside the real numbers. By [Jar11, Example 5.10.7], the field $\mathbb{Q}^{\mathrm{tot}\mathbb{R}}(\sqrt{-1})$ is an example of a PAC field. From the results contained in this paper, it is easy to obtain an explicit $\mathcal{L}_{\mathrm{ring}}$-formula which defines $\mathcal{O}_w$ in the field

$$K = \mathbb{Q}^{\mathrm{tot}\mathbb{R}}(\sqrt{-1})((\Gamma_1))((\Gamma_2))$$

and which – by Theorem 1.3 – is not equivalent to a $\emptyset$-$\forall\exists$-formula or a $\emptyset$-$\exists\forall$-formula modulo $\mathrm{Th}(K)$.

Note that in all examples constructed, w admits proper henselian refinements and hence is *not* the canonical henselian valuation of K. Thus, our results do not contradict Theorem 1.1 in [FJ15] which states that the canonical henselian valuation is in most cases $\emptyset$-$\forall\exists$-definable or $\emptyset$-$\exists\forall$-definable as soon as it is $\emptyset$-definable at all (see also [FJ15] for the definition of the canonical henselian valuation of a field).

2. The construction

2.1. **The value group.** In this section, we consider examples of (Hahn) sums of ordered abelian groups. For H and G ordered abelian groups, consider the lexicographic sum $G \oplus H$, that is the ordered group with underlying set $G \times H$ and equipped with the lexicographic order such that G is more significant. More generally, recall that for a totally ordered set $(I, <)$ and a family $(G_i)_{i\in I}$ of ordered abelian groups, there is a corresponding Hahn sum

$$G := \bigoplus_{i\in I} G_i.$$

consisting of all sequences $(g_i)_{i\in I} \in \prod_{i\in I} G_i$ with finite support. Componentwise addition and the lexicographic order (where G_i is more significant than $G_{i'}$ if $i < i'$) give G the structure of an ordered abelian group. For any $k \in I$, the final segment $\bigoplus_{i\in I,\, i>k} G_i$ is a convex subgroup of G and the quotient of G by said subgroup is isomorphic to the corresponding initial segment $\bigoplus_{i\in I,\, i\leq k} G_i$.

We consider the ordered abelian groups

$$X := \mathbb{Z}_{(2)} = \left\{\frac{a}{b} \in \mathbb{Q} \,|\, a, b \in \mathbb{Z},\ 2 \nmid b\right\} \text{ and } Y := \mathbb{Z}_{(3)} = \left\{\frac{a}{b} \in \mathbb{Q} \,|\, a, b \in \mathbb{Z},\ 3 \nmid b\right\}$$

as building blocks in the construction of Hahn sums. All ordered abelian groups considered in this note are of the form $\bigoplus_{j\in J} G_j$ for some ordered index set J with $G_j \in \{X, Y\}$ for all $j \in J$. Let $(\mathbb{N}, <)$ denote the natural numbers with their usual ordering and $(\mathbb{N}', <)$ the natural numbers in reverse order. Define

$$\Gamma_1 := \bigoplus_{\mathbb{N}}((\bigoplus_{\mathbb{N}} Y) \oplus X)$$

and

$$\Gamma_2 := \bigoplus_{\mathbb{N}'}(X \oplus Y).$$

Then, the ordered abelian group Y is the quotient of Γ_1 by its convex subgroup

$$\Lambda_1 := ((\bigoplus_{\mathbb{N}\setminus\{0\}} Y) \oplus X)) \oplus (\bigoplus_{\mathbb{N}\setminus\{0\}}((\bigoplus_{\mathbb{N}} Y) \oplus X)).$$

Note that there is an isomorphism $f_1 : \Lambda_1 \xrightarrow{\sim} \Gamma_1$ of ordered abelian groups induced by the (unique) isomorphism of the index sets. Furthermore, $X \oplus Y$ is a convex subgroup

of Γ_2, with corresponding quotient

$$\Lambda_2 = \bigoplus_{\mathbb{N}'\setminus\{0\}} (X \oplus Y).$$

Again, the (unique) isomorphism of the index sets induces an isomorphism $g_2 : \Lambda_2 \xrightarrow{\sim} \Gamma_2$. We now consider the lexicographic sum

$$\Gamma := \Gamma_2 \oplus \Gamma_1.$$

Lemma 2.1. *Let Γ be as above. Then, the convex subgroup Γ_1 is a parameter-free $\mathcal{L}_{\mathrm{oag}}$-definable subgroup of Γ.*

Proof. We write Γ as a Hahn sum

$$\Gamma = \bigoplus_{j\in J} G_j$$

with $G_j \in \{X, Y\}$. There is a smallest element $k \in J$ which has a successor k' such that $G_k = G_{k'} = Y$. For that k, one has

$$\Gamma_1 = \bigoplus_{j\in J,\, j>k} G_j;$$

the idea of this proof is to express this as a formula, using that J is interpretable in Γ; see e.g. [CH11] or [Sch82]. We now explain this interpretation in some detail.

Fix $r \in \mathbb{N}$ (we will only consider $r = 3, 6$). For $x \in \Gamma\setminus r\Gamma$, let $F_r(x)$ be the largest convex subgroup of Γ which is disjoint from $x + r\Gamma$. For fixed r, $F_r(x)$ is definable uniformly in x by [Sch82, Lemma 2.11] or [CH11, Lemma 2.1], namely:

$$y \in F_r(x) \iff [0, r\max\{-y, y\}] \cap x + r\Gamma = \emptyset.$$

Using that all G_j are archimedean, one can check that the set of groups of the form $F_r(x)$ ($x \in \Gamma \setminus r\Gamma$) is exactly equal to the set of groups of the form

$$\bigoplus_{j\in J, j>j_0} G_j,$$

where j_0 runs over those $j \in J$ for which G_j is not r-divisible; see [Sch82, Example 2.3] or a combination of the examples in [CH11, Sections 4.1 and 4.2] for details.

Thus we have the interpretation $J = (\Gamma \setminus 6\Gamma)/{\sim_6}$, where $x \sim_r x'$ iff $F_r(x) = F_r(x')$, and

$$J_Y := (\Gamma \setminus 3\Gamma)/{\sim_3} = \{j \in J \mid G_j = Y\}.$$

Now our k from above is a $\emptyset$-definable element of J and we have $F_6(x) = \Gamma_1$ for any $x \in \Gamma \setminus 6\Gamma$ with $x/{\sim_6} = k$, as desired. $\square$

Next, we give different existentially closed embeddings of Γ into itself which we will use to apply Prestel's Theorem. We use the following facts:

Theorem 2.2 ([Wei90, Corollaries 1.4 and 1.7])**.** *Let G_1 and G_2 be ordered abelian groups.*

1. *If G_1 is a convex subgroup of G_2, then G_1 is existentially closed in G_2.*
2. *Consider the Hahn sum $G = G_2 \oplus G_1$. Let G_1' (resp. G_2') be an ordered subgroup of G_1 (resp. G_2) that is existentially closed in G_1 (resp. G_2), and put $G' := G_2' \oplus G_1'$. Then G' is existentially closed in G.*

The first embedding $f_3 : \Gamma \to \Gamma$ which we want to consider is given by $f_1 : \Lambda_1 \to \Gamma_1$ (defined above) and $f_2 : \Gamma_2 \oplus Y \to \Gamma_2$ which maps Γ_2 isomorphically to Λ_2 via g_2^{-1} (defined above) and which embeds Y into $X \oplus Y$ as a convex subgroup:

$$\begin{aligned} f_3 : \Gamma_2 \oplus \Gamma_1 = \Gamma_2 \oplus Y & \qquad \oplus \Lambda_1 \\ \cong \underbrace{f_2(\Gamma_2 \oplus Y)}_{\prec_\exists \Gamma_2} & \qquad \oplus \underbrace{f_1(\Lambda_1)}_{=\Gamma_1} \\ \prec_\exists \Gamma_2 & \qquad \oplus \Gamma_1 \end{aligned} \tag{2.1}$$

The second embedding is $g_3 : \Gamma \to \Gamma$ given by $g_2 : \Lambda_2 \to \Gamma_2$ (defined above) and $g_1 : (X \oplus Y) \oplus \Gamma_1 \to \Gamma_1$ which embeds it as a convex subgroup. More precisely, we consider the isomorphism

$$g_{1,1} : \Gamma_1 \xrightarrow{\sim} ((\bigoplus_{\mathbb{N}\setminus\{0\}} Y) \oplus X) \oplus \bigoplus_{\mathbb{N}\setminus\{0,1\}} (\bigoplus_{\mathbb{N}} Y) \oplus X$$

induced by the (unique) order isomorphism of the index sets, and the embedding

$$g_{1,2} : X \oplus Y \to (\bigoplus_{\mathbb{N}} Y) \oplus X \oplus Y$$

as a convex subgroup which maps $X \oplus Y$ onto itself as a final segment of the Hahn sum on the right. Overall, we obtain the following embedding of Γ into itself:

$$\begin{aligned} g_3 : \Gamma_2 \oplus \Gamma_1 = \Lambda_2 & \qquad \oplus (X \oplus Y) \oplus \Gamma_1 \\ \cong \underbrace{g_2(\Lambda_2)}_{=\Gamma_2} & \qquad \oplus \underbrace{g_1((X \oplus Y) \oplus \Gamma_1)}_{\prec_\exists \Gamma_1} \\ \prec_\exists \Gamma_2 & \qquad \oplus \Gamma_1 \end{aligned} \tag{2.2}$$

2.2. **The residue field.** Let k be a PAC field which is not separably closed. Then, any henselian valuation with residue field k is $\emptyset$-definable ([JK15, Lemma 3.5 and Theorem 3.6]). Moreover, assume that k is a PAC field of characteristic 0 such that the algebraic part k_0 of k is not algebraically closed, i.e., $k_0 := \mathbb{Q}^{alg} \cap k \subsetneq \mathbb{Q}^{alg}$. By [Feh15, Theorem 3.5 and its proof], any henselian valuation with residue field k is $\emptyset$-$\exists$-definable: In fact, for any monic and irreducible $f \in k_0[X]$ with $\deg(f) > 1$, [Feh15, Section 3] gives a parameter-free $\mathcal{L}_{\text{ring}}$-formula depending on f which defines the valuation ring of v in any henselian valued field (K, v) with residue field k.

In order to get an explicit example, we consider the maximal totally real extension $\mathbb{Q}^{\text{tot}\mathbb{R}}$ of $\mathbb{Q}$. As mentioned in the introduction, $k := \mathbb{Q}^{\text{tot}\mathbb{R}}(\sqrt{-1})$ is a PAC field by [Jar11, Example 5.10.7]. Furthermore, as $\sqrt[3]{2}$ is not totally real, $f = X^3 - 2$ is a monic and irreducible polynomial with coefficients in the algebraic part k_0 of k. Thus, by [Feh15, Proposition 3.3], the formula

$$\begin{aligned} \eta(x) \equiv (\exists u, t)(x = u + t \wedge (\exists y, z, y_1, z_1)(u = y_1 - z_1 \wedge y_1(y^3 - 2) = 1 \wedge z_1(z^3 - 2) = 1) \\ \wedge (\exists y, z, y_1, z_1)(t = 0 \vee (t = y_1 z_1 \wedge y_1(y^3 - 2) = 1 \wedge z_1(z^3 - 2) = 1)) \end{aligned}$$

defines the valuation ring of v in any henselian valued field (K, v) with residue field k.

2.3. **Power series fields.** Now, define $K := k((\Gamma_1))((\Gamma_2)) = k((\Gamma_2 \oplus \Gamma_1))$ for k PAC but not separably closed. Then, the valuation ring of the henselian valuation v on K with value group $\Gamma_2 \oplus \Gamma_1$ and residue field k is $\emptyset$-definable by the results discussed in the previous section. Moreover, for $k = \mathbb{Q}^{\text{tot}\mathbb{R}}$, $\mathcal{O}_v$ is $\emptyset$-$\exists$-definable by the formula $\eta(x)$ (as above). Let w be the coarsening of v with value group Γ_2 and residue field $k((\Gamma_1))$.

Recall that by Lemma 2.1, the convex subgroup Γ_1 is $\emptyset$-definable in the ordered abelian group $\Gamma_2 \oplus \Gamma_1$. Thus, w is $\emptyset$-definable on K.

We now give two different existentially closed embeddings of K into itself which combined with Prestel's Characterization Theorem show that w is neither $\emptyset$-$\forall\exists$-definable nor $\emptyset$-$\exists\forall$-definable.

Theorem 2.3 (Ax-Kochen/Ersov, see [KP84, p. 183])**.** *Let (K, w) be a henselian valued field of equicharacteristic 0. Let $(K, w) \subseteq (L, u)$ be an extension of valued fields. If the residue field of (K, w) is existentially closed in the residue field of (L, u) and the value group of (K, w) is existentially closed in the value group of (L, u), then (K, w) is existentially closed in (L, u).*

Construction 2.4. *Let $K = k((\Gamma_1))((\Gamma_2))$ with Γ_1 and Γ_2 as before. Let w denote the power series valuation on K with valuation ring $k((\Gamma_1))[[\Gamma_2]]$ and value group Γ_2.*

(1) *Consider the existential embeddings $f_0 = \mathrm{id}_k$, as well as f_3 as defined in Equation (2.1). By Theorem 2.3, there is an existential embedding $f : K \to K$ which prolongs f_0 and f_3. Then, as the embedding maps more than just Γ_2 into Γ_2, we have $f(\mathcal{O}_w) \supsetneq \mathcal{O}_w$.*

(2) *On the other hand, consider the existential embeddings $g_0 = \mathrm{id}_k$, as well as g_3 as defined in Equation (2.2). Once again, there is an existential embedding $g : K \to K$ which prolongs g_0 and g_3. Then, as the embedding maps more than just Γ_1 into Γ_1, we have $g(\mathcal{O}_w) \subsetneq \mathcal{O}_w$.*

In particular, the henselian valuation w with value group Γ_2 is $\emptyset$-definable on

$$K = \mathbb{Q}^{\mathrm{tot}\mathbb{R}}(\sqrt{-1})((\Gamma_1))((\Gamma_2))$$

but neither $\emptyset$-$\forall\exists$-definable nor $\emptyset$-$\exists\forall$-definable by Theorem 1.1. This finishes the proof of Theorem 1.3.

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